MIXED COMMUTING VARIETIES

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ABSTRACT. Let $\mathfrak{g}$ be a simple Lie algebra defined over an algebraically closed field k of characteristic p. Fix an integer $r > 1$ and suppose that $V_1, \ldots, V_r$ are irreducible closed subvarieties of $\mathfrak{g}$. Let $C(V_1, \ldots, V_r)$ be the closed variety of all the pairwise commuting elements in $V_1 \times \cdots \times V_r$. This paper is aimed to study the dimension and irreducibility of the variety $C(V_1, \ldots, V_r)$ with various V_i's in a Lie algebra $\mathfrak{g}$ and show its applications to representation theory. In particular, we first show that the variety $C_r(z_{\mathrm{sub}})$, where z_{sub} is the centralizer of a subregular element in $\mathfrak{sl}_n$, is a direct product of an affine space and a determinantal variety. This result is applied to give an alternative computation for the dimensions of $C_r(\mathfrak{sl}_3)$ and $C_r(\mathfrak{gl}_3)$. Next we calculate the dimension of every mixed commuting variety over $\overline{\mathcal{O}_{\mathrm{sub}}}$, $\mathcal{N}$ and $\mathfrak{sl}_3$ and determine whether it is irreducible or not. Finally, we apply our calculations to study properties of support varieties for a simple module of the r-th Frobenius kernels of SL_3.

1. INTRODUCTION

1.1. Let k be an algebraically closed field of characteristic p (possibly $p = 0$). Suppose $\mathfrak{g}$ is a simple Lie algebra defined over k. For each $r \geq 2$, let $V_1, \ldots, V_r$ be irreducible closed subvarieties of a Lie algebra $\mathfrak{g}$. Define

$$C(V_1, \ldots, V_r) = \{(v_1, \ldots, v_r) \in V_1 \times \cdots \times V_r \mid [v_i, v_j] = 0 , \ 1 \leq i \leq j \leq r\},$$

a *mixed commuting variety* over $V_1, \ldots, V_r$[1]. Note that if $V_1 = \cdots = V_r$, then this variety becomes $C_r(V_1)$, the commuting variety of r-tuples. Mixed commuting varieties of two tuples was first introduced in [Vas, 9.4], the r-tuple version was defined and studied in [N]. In particular, the author explicitly describe the irreducible decomposition for any mixed commuting variety over $\mathfrak{sl}_2$ and its nullcone $\mathcal{N}$. The result also implies that such varieties are mostly not Cohen-Macaulay or normal.

In general, mixed commuting varieties are still mysterious. Interesting questions include

(1) What is the dimension of $C_r(V_1, \ldots, V_r)$?
(2) What are the irreducible components of $C_r(V_1, \ldots, V_r)$?

The results in this paper were motivated by investigating the cohomology for Frobenius kernels of algebraic groups. The first connection between commuting varieties and support varieties in cohomology of Frobenius kernels was constructed by Suslin, Friedlander, and Bendel in their two papers [SFB1, SFB2]. To be precise, let G be an algebraic group defined over k, and let G_r be the r-th Frobenius kernel of G. Then there is a homeomorphism between the maximal ideal spectrum of the cohomology ring for G_r and the nilpotent commuting variety over the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ whenever the characteristic p is large enough. This variety is also the ambient space for support varieties of the r-th Frobenius kernels of G. In the case $r = 1$, these support varieties for modules $L(\lambda)$ and $\mathrm{H}^0(\lambda)$ are explicitly described by the work Drupieski, Nakano, Parshall, and Vella [NPV][DNP]. Sobaje uses these desriptions to compute the support varieties of $L(\lambda)$ for higher r [So]. Our study on mixed commuting varieties is inspired from results in this paper [So, Theorems 3.1, 3.2]. Most of our calculations about mixed commuting varieties are in the case

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[1]We are aware of the same terminology arisen in [Ya2, Section 4]. The definition of mixed commuting varieties in this paper is entirely different with ours.

$\mathfrak{g} = \mathfrak{sl}_3$. More explicitly, we first study the dimensions of the commuting varieties over $\mathfrak{gl}_3$, $\mathfrak{sl}_3$. These are not new results, however we use a new method to tackle the problem. Our approaching does not depend on the irreducibility of $C_r(\mathfrak{gl}_3)$. On the other hand, we investigate the dimension and reducibility of mixed commuting varieties over $\overline{\mathcal{O}}_{\text{sub}}$, $\mathcal{N}$ and $\mathfrak{sl}_3$. Our results indicates that these mixed commuting varieties are mostly not irreducible. Hence, they are rarely normal.

1.2. **Main results.** The paper is organized as follows. We first review terminology and notation in Section 2. Then in Section 3, we study the properties of $C_r(z_{\text{sub}})$. In particular, let $\mathfrak{g} = \mathfrak{sl}_n$ and $p \nmid n$ then we prove that for every $r \geq 1$, $C_r(z_{\text{sub}})$ is a product of an affine space of dimension $(n-2)r$ with a determinantal variety generated by all 2×2-minors over a $(3 \times r)$-matix of indeterminants; hence it is normal and Cohen-Macaulay (cf. Theorem 3.1.1). Next, we apply our calculations to compute the dimensions of $C_r(\mathfrak{sl}_3)$ and $C_r(\mathfrak{gl}_3)$. Notice that the latter variety was proved to be irreducible by Kirillov and Neretin [KN], see also [Gu]; hence the dimension easily follows. Our method does not depend on the irreducibility of this variety. We also emphasize the impact of the characteristic p, i.e. $p \mid n$, on the commuting variety $C_r(\mathfrak{sl}_3)$ (cf. Remark 3.2.3).

In Section 4, we investigate mixed commuting varieties over $\overline{\mathcal{O}}_{\text{sub}}$, $\mathcal{N}$, and $\mathfrak{sl}_3$. Our key ingredient is determinantal varieties over certain matrices. In particular, by analyzing the intersections $z_{\text{sub}} \cap \overline{\mathcal{O}}_{\text{sub}}$ and $z_{\text{sub}} \cap \mathcal{N}$, we reduce to the problem of computing the dimension of the varieties generated by 2-minors of the following matrix of indeterminants

$$\begin{pmatrix} x_1 & \cdots & x_i & x_{i+1} & \cdots & x_{i+j} & x_{i+j+1} & \cdots & x_{i+j+m} \\ 0 & \cdots & 0 & y_1 & \cdots & y_j & y_{j+1} & \cdots & y_{j+m} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & z_1 & \cdots & z_m \end{pmatrix}.$$

By checking all possibilities, we figure out the dimension formula for these varieties depending on parameters i, j, m (cf. Theorem 4.3.4). This observation implies the dimension formula for mixed commuting varieties. Moreover, we are able to determine whether a mixed commuting variety is irreducible or not (cf. Theorems 4.4.1 and 4.4.3).

In the last section, we use results in the previous section to compute the dimension of support varieties of Frobenius kernels for a simple module $L(\lambda)$ with $p > 6$. In particular, let $G = SL_3$ and $\lambda \in X^+$. Suppose that

$$\lambda = \lambda_0 + \lambda_1 p + \cdots + \lambda_q p^q$$

with $\lambda_i \in X_1$. We compute the dimension of the support variety $V_{G_r}(L(\lambda))$ as follows.

$$\dim V_{G_r}(L(\lambda)) = \begin{cases} 2b_\lambda + 4 & \text{if } a_\lambda = 0, \\ 2(a_\lambda + b_\lambda) + 3 & \text{if } a_\lambda = 1, \\ 2(a_\lambda + b_\lambda) + 2 & \text{if } a_\lambda > 1. \end{cases}$$

where a_λ, b_λ be the number of singular weights and regular weights respectively in $\{\lambda_1, \ldots, \lambda_q\}$. We further show that $V_{G_r}(L(\lambda))$ is always reducible unless every λ_i is regular (cf. Theorems 5.1.3 and 5.1.4).

2. Notation

2.1. **Root systems and combinatorics.** Let k be an algebraically closed field of characteristic p. Let G be a simple, simply-connected algebraic group over k, defined and split over the prime field $\mathbb{F}_p$. Fix a maximal torus $T \subset G$, also split over $\mathbb{F}_p$, and let Φ be the root system of T in G. Fix a set $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ of simple roots in Φ, and let Φ^+ be the corresponding set of positive roots. Let $B \subseteq G$ be the Borel subgroup of G containing T and corresponding to the set of negative roots Φ^-, and let $U \subseteq B$ be the unipotent radical of B. Set $\mathfrak{g} = \text{Lie}(G)$, the Lie algebra of G, $\mathfrak{b} = \text{Lie}(B)$, $\mathfrak{u} = \text{Lie}(U)$.

Let X be the weight lattice of Φ. Write X^+ for the set of dominant weights in X, and X_r for the set of p^r-restricted dominant weights in X^+. Given $\lambda \in X^+$, let $L(\lambda)$ be the simple rational

G-module of highest weight λ. For each $r \geq 1$, let $F^r : G \to G$ be the r-th iterate Frobenius morphism of G. We call $G_r = \ker F_r$ the r-Frobenius kernel of G.

2.2. Nilpotent orbits.

Given a G-variety V and a point v of V, we denote by $\mathcal{O}_v$ the G-orbit of v (i.e., $\mathcal{O}_v = G \cdot v$). For example, consider the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$ as a G-variety with the adjoint action. There are well-known orbits: $\mathcal{O}_{\mathrm{reg}} = G \cdot v_{\mathrm{reg}}, \mathcal{O}_{\mathrm{subreg}} = G \cdot v_{\mathrm{subreg}}$, (we abbreviate it by $\mathcal{O}_{\mathrm{sub}}$,) and $\mathcal{O}_{\mathrm{min}} = G \cdot v_{\mathrm{min}}$ where $v_{\mathrm{reg}}, v_{\mathrm{subreg}}$, and v_{min} are representatives for the regular, subregular, and minimal orbits. Denote by $z(v)$ the centralizer of v in $\mathfrak{g}$. For convenience, we write z_{reg} (z_{sub} and z_{min}) for the centralizers of v_{reg} (v_{sub} or v_{min}).

2.3. Basic algebraic geometry conventions.

Let R be a commutative Noetherian ring with identity. We use R_{red} to denote the reduced ring $R/\sqrt{0}$ where $\sqrt{0}$ is the radical ideal of the trivial ideal 0, which consists of all nilpotent elements of R. Let $\operatorname{Spec} R$ be the spectrum of all prime ideals of R. If V is a closed subvariety of an affine space $\mathbb{A}^n$, we denote by $I(V)$ the radical ideal of $k[\mathbb{A}^n] = k[x_1, \ldots, x_n]$ associated to this variety. Let X be an affine variety. Then we always write $k[X]$ for the coordinate ring of X which is the same as the ring of global sections $\mathcal{O}_X(X)$.

2.4. Commutative algebra.

Consider an $m \times n$ matrix

$$
X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix}
$$

whose entries are independent indeterminates over the field k. Let $k(X)$ be the polynomial ring over all the indeterminates of X, and let $I_t(X)$ be the ideal in $k(X)$ generated by all t-minors of X. For each $t \geq 1$, the ring

$$
R_t(X) = \frac{k(X)}{I_t(X)}
$$

is called a *determinantal ring*. The following is one of the nice properties of determinantal rings.

Proposition 2.4.1. [BV] *For every* $1 \leq t \leq \min(m, n)$, *$R_t(X)$ is a reduced, Cohen-Macaulay, normal domain of dimension* $(t - 1)(m + n - t + 1)$.

We denote by $D_t(X)$ the determinantal variety defined by $I_t(X)$.

3. COMMUTING VARIETIES OF CENTRALIZERS

Let $\mathfrak{g} = \mathfrak{sl}_n$. It is easy to see that $C_r(z_{\mathrm{reg}}) = z_{\mathrm{reg}}^r$ for every $r \geq 1$. We study in this section the variety $C_r(z_{\mathrm{sub}})$. Then we apply our calculations to compute the dimensions of $C_r(\mathfrak{sl}_3)$ and $C_r(\mathfrak{gl}_3)$ for each $r \geq 1$.

3.1. Nice properties of $C_r(z_{\mathrm{sub}})$.

Theorem 3.1.1. *For each $r \geq 1$, the variety $C_r(z_{\mathrm{sub}})$ is irreducible, Cohen-Macaulay and normal. Moreover, we have*

$$
\dim C_r(z_{\mathrm{sub}}) = \begin{cases} (n - 1)r + 2 & \text{if } p \nmid n, \\ nr + 1 & \text{otherwise.} \end{cases}
$$

Proof. Without loss of generality, let v_{sub} be the Jordan matrix corresponding to the partition $[n-1, 1]$. Then an element u of z_{sub} is of the form

$$u = \begin{pmatrix} a_1 & 0 & 0 & \cdots & \cdots & 0 \\ a_2 & a_1 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ a_{n-1} & \cdots & \cdots & a_2 & a_1 & c \\ b & 0 & \cdots & \cdots & 0 & (1-n)a_1 \end{pmatrix}.$$

By using the multiplication of matrices by blocks, we obtain for any pair u, u' in z_{sub}

$$[u, u'] = \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ cb' - bc' & \cdots & \cdots & 0 & 0 & n(a_1 c' - a_1' c) \\ n(ba_1' - a_1 b') & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}.$$

If p does not divide n, the defining polynomials of the commutator are $a_1 c' - a_1' c, cb' - bc'$, and $ba_1' - b'a_1$. This implies that the variety $C_r(z_{\text{sub}})$ is defined by the collection of polynomials $\{x_i y_j - x_j y_i \,,\, y_i z_j - y_j z_i \,,\, x_i z_j - x_j z_i \mid 1 \le i \le j \le r\}$ in $k[x_i, y_i, z_i \mid 1 \le i \le r]$. So we can identify $C_r(z_{\text{sub}})$ with the determinantal variety $D_2(X)$ where X is the matrix

$$X = \begin{pmatrix} x_1 & x_2 & \cdots & x_r \\ y_1 & y_2 & \cdots & y_r \\ z_1 & z_2 & \cdots & z_r \end{pmatrix}.$$

This identification implies the following isomorphism of varieties

$$C_r(z_{\text{sub}}) \cong D_2(X) \times \mathbb{A}^{(n-2)r}.$$

On the other hand, if p divides n, then the commutator is defined by $cb' - c'b$. This gives us the following

$$C_r(z_{\text{sub}}) = D_2(Y) \times \mathbb{A}^{(n-1)r}$$

where Y is the matrix of indeterminates defined by

$$Y = \begin{pmatrix} x_1 & x_2 & \cdots & x_r \\ y_1 & y_2 & \cdots & y_r \end{pmatrix}.$$

Hence $D_2(Y)$ is of dimension $r + 1$ and so we obtain $\dim C_r(z_{\text{sub}}) = (n-1)r + r + 1 = nr + 1$. Other results immediately follow from Proposition 2.4.1. $\qquad\square$

Remark 3.1.2. This decomposition of $C_r(z_{\text{sub}})$ is compatible with the result of Neubauer and Sethuraman on commuting pairs in the centralizers of 2-regular matrices when $r = 2$ [NS]. This result can not be generalized to the centralizer of arbitrary nilpotent element. In particular, there is a nilpotent element e in $\mathfrak{g}$ such that $C_2(z(e))$ is reducible (hence so is $C_r(z(e))$) [Ya1].

3.2. **Application.** Suppose $\mathfrak{g}$ is an arbitrary simple Lie algebra. We first study a connection between the dimension of the commuting variety $C_r(\mathfrak{g})$ and that of a certain mixed commuting variety.

Define a mixed commuting variety by

$$C(\mathcal{N}, \mathfrak{g}^{r-1}) = \{(v_1, \ldots, v_r) \in \mathcal{N} \times \mathfrak{g}^{r-1} \mid [v_i, v_j] = 0 \text{ with } 1 \le i \le j \le r\},$$

which is a subvariety of $C_r(\mathfrak{g})$ with nilpotency condition for the first factor. Note that

(1) $\dim C_r(\mathfrak{g}) \le \dim C(\mathcal{N}, \mathfrak{g}^{r-1}) + \text{rank}(\mathfrak{g})$.

Hence, the dimension of $C(\mathcal{N}, \mathfrak{g}^{r-1})$ gives an upper bound of $\dim C_r(\mathfrak{g})$. Note also that in the case $r = 2$, Baranovsky used this variety to compute the dimension of $C_2(\mathcal{N})$, [Ba, Theorem 2]. Assume in this subsection that $p \neq 3$. We aim to compute the dimensions of $C_r(\mathfrak{sl}_3)$ and $C_r(\mathfrak{gl}_3)$. We begin with a lemma.

Theorem 3.2.1. *For each $r \geq 2$, we have $\dim C(\mathcal{N}, \mathfrak{sl}_3^{r-1}) = 2r + 4$ and $\dim C_r(\mathfrak{sl}_3) = 2r + 6$.*

Proof. As $C_r(\mathfrak{sl}_3)$ contains a component $\overline{G \cdot \mathfrak{t}^r}$, where $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{sl}_3$. It is easy to see that the dimension of this component is $2r + 6$. So it suffices to show that $\dim C_r(\mathfrak{sl}_3) \leq 2r + 6$. We process by induction and assume that $\dim C_{r-1}(\mathfrak{sl}_3) \leq 2(r-1) + 6 = 2r + 4$. As $\mathcal{N}$ contains three nilpotent orbits: $\mathcal{O}_{\mathrm{reg}}, \mathcal{O}_{\mathrm{sub}}$, and 0, we have the following decomposition

$$(2) \qquad C(\mathcal{N}, \mathfrak{sl}_3^{r-1}) = \overline{G \cdot (v_{\mathrm{reg}}, z_{\mathrm{reg}}, \dots, z_{\mathrm{reg}})} \cup \overline{G \cdot (v_{\mathrm{sub}}, C_{r-1}(z_{\mathrm{sub}}))} \cup 0 \times C_{r-1}(\mathfrak{sl}_3).$$

in which the dimension of each component is given as follows:

$$\dim \overline{G \cdot (v_{\mathrm{reg}}, z_{\mathrm{reg}}, \dots, z_{\mathrm{reg}})} = \dim G \cdot v_{\mathrm{reg}} + \dim z_{\mathrm{reg}}^{r-1} = 6 + 2(r-1) = 2r + 4,$$
$$\dim \overline{G \cdot (v_{\mathrm{sub}}, C_{r-1}(z_{\mathrm{sub}}))} = \dim G \cdot v_{\mathrm{sub}} + \dim C_{r-1}(z_{\mathrm{sub}}) = 4 + 2(r-1) + 2 = 2r + 4$$

where $\dim C_{r-1}(z_{\mathrm{sub}}) = 2(r-1) + 2$ by Theorem 3.1.1. So the dimension of $C(\mathcal{N}, \mathfrak{sl}_3^{r-1})$ is $2r + 4$, which confirms the first statement. Then the inequality (1) implies that

$$\dim C_r(\mathfrak{sl}_3) \leq 2r + 4 + \mathrm{rank}\,\mathfrak{sl}_3 = 2r + 6$$

which completes our proof. $\qquad\square$

Corollary 3.2.2. *For each $r \geq 2$, we have $\dim C_r(\mathfrak{gl}_3) = 3r + 6$.*

Proof. It follows from Theorem 3.2.1 and [N, Theorem 4.2.1]. $\qquad\square$

Remark 3.2.3. Note first that our computation does not reply on the irreducibility of $C_r(\mathfrak{gl}_3)$ for each $r \geq 1$. In the case $p = 3$, Remark 3.1.2 shows that $C_r(\mathfrak{sl}_3)$ is of dimension at least $3r + 2$. This shows that $C_r(\mathfrak{sl}_3)$ is reducible when $r > 4$.

4. Mixed Commuting Varieties

We apply in this section a new technique to compute the dimension of mixed commuting varieties over various closed sets in $\mathfrak{sl}_3$. Our calculations are based on the dimension for a certain class of varieties defined by minors of a matrix of indeterminates.

To begin we set

$$C_{i,j,m} = C(\underbrace{\overline{\mathcal{O}_{\mathrm{sub}}}, \dots, \overline{\mathcal{O}_{\mathrm{sub}}}}_{i \text{ times}}, \underbrace{\mathcal{N}, \dots, \mathcal{N}}_{j \text{ times}}, \underbrace{\mathfrak{sl}_3, \dots, \mathfrak{sl}_3}_{m \text{ times}}).$$

Our goal is to compute the dimension of $C_{i,j,m}$ for every non-negative integers i, j, m.

4.1. Note that we have known $\dim C_{i,j,m}$ in the following cases:
- If $i = j = 0$ then $\dim C_{i,j,m} = \dim C_m(\mathfrak{sl}_3) = 2m + 6$,
- If $i = m = 0$ then $\dim C_{i,j,m} = \dim C_j(\mathcal{N}) = 2j + 4$,
- If $j = m = 0$ then $\dim C_{i,j,m} = \dim C_i(\overline{\mathcal{O}}_{\mathrm{sub}}) = 2i + 2$

by Theorem 3.2.1 and [N, Theorems 7.1.2 and 7.2.3]. When $i = 0$, the dimension of $C_{i,j,m}$ can be easily computed as follows.

Proposition 4.1.1. *For $j, m \geq 1$, we have $\dim C_{0,j,m} = 2(j + m) + 4$. Therefore, the variety $C_{0,j,m}$ is never irreducible.*

Proof. Observe that

$$\dim C_{j+m}(\mathcal{N}) \le \dim C_{0,j,m} \le \dim C(\mathcal{N}, \mathfrak{sl}_3^{j+m-1}).$$

From earlier we have

$$2(j+m) + 4 \le \dim C_{0,j,m} \le 2(j+m) + 4.$$

This gives us the dimension of $C_{0,j,m}$. It indicates that $C_{i+m}(\mathcal{N})$ is a proper irreducible component of $C_{0,j,m}$. Hence, the reducibility of $C_{0,j,m}$ is proved. $\square$

4.2. Fix v_{sub}, the canonical Jordan block matrix corresponding to the partition $[2,1]$ of 3. Then the centralizer of v_{sub} in $\mathfrak{sl}_3$ is

$$\left\{ \begin{pmatrix} x & 0 & 0 \\ y & x & t \\ z & 0 & -2x \end{pmatrix} \mid x,y,z,t \in k \right\}.$$

We recall results in [N] on the intersections of z_{sub} with $\overline{\mathcal{O}_{\mathrm{sub}}}$ or $\mathcal{N}$ respectively which play important roles in our calculations.

Proposition 4.2.1. [N, Lemma 7.2.2] *There are identities*

$$z_{\mathrm{sub}} \cap \mathcal{N} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ y & 0 & t \\ z & 0 & 0 \end{pmatrix} \mid y,z,t \in k \right\},$$

$$z_{\mathrm{sub}} \cap \overline{\mathcal{O}_{\mathrm{sub}}} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ y & 0 & 0 \\ z & 0 & 0 \end{pmatrix} \cup \begin{pmatrix} 0 & 0 & 0 \\ y & 0 & t \\ 0 & 0 & 0 \end{pmatrix} \mid y,z,t \in k \right\}$$

$$= V_1 \cup V_2.$$

Moreover, if $u,v \in V_1 \cup V_2$ then

$$[u,v] = 0 \Leftrightarrow u,v \in V_1 \quad \text{or} \quad u,v \in V_2.$$

4.3. **Some results in determinantal varieties.** Before investigating the dimension of the mixed commuting variety $C_{i,j,m}$, we need to prove some results related to dimensions of determinantal varieties.

Lemma 4.3.1. *Let X be the matrix of indeterminants*

$$\begin{pmatrix} x_1 & \cdots & x_i & x_{i+1} & \cdots & x_{i+j} \\ 0 & \cdots & 0 & y_1 & \cdots & y_j \end{pmatrix}.$$

Then the dimension of $V(I_2(X))$ is $j+1$ if $i=0$ and $i+j$ otherwise.

Proof. It is easy to see that if $i=0$, then $\dim V(I_2(X)) = j+1$ by Proposition 2.4.1. Otherwise, we note that

$$I_2(X) = I_2(X') + \langle y_1', \ldots, y_i' \rangle$$

where

$$X' = \begin{pmatrix} x_1 & \cdots & x_i & x_{i+1} & \cdots & x_{i+j} \\ y_1' & \cdots & y_i' & y_1 & \cdots & y_j \end{pmatrix}.$$

As the ideal $I_2(X')$ is prime and y_1' does not belong to $I_2(X')$, we have

$$\dim V(I_2(X)) \le \dim V(I_2(X')) - 1 = i + j.$$

On the other hand, since $V(I_2(X))$ contains an affine space of dimension $i+j$, we must have $\dim V(I_2(X)) = i + j$. $\square$

Lemma 4.3.2. *Let Y be the matrix of indeterminants*

$$\begin{pmatrix} x_1 & \cdots & x_i & x_{i+1} & \cdots & x_{i+j} \\ y_1 & \cdots & y_i & y_{i+1} & \cdots & y_{i+j} \\ 0 & \cdots & 0 & z_1 & \cdots & z_j \end{pmatrix}.$$

Then the dimension of $V(I_2(Y))$ is $j+2$ if $i=0$ and $i+j+1$ otherwise.

Proof. The case $i=0$ follows immediately by Proposition 2.4.1. Suppose $i > 0$, we have

$$\dim V(I_2(Y')) = i+j+1 \le \dim V(I_2(Y)) \le \dim V(I_2(Y'')) - 1$$

where Y' and Y'' are the following matrices

$$\begin{pmatrix} x_1 & \cdots & x_i & x_{i+1} & \cdots & x_{i+j} \\ y_1 & \cdots & y_i & y_{i+1} & \cdots & y_{i+j} \end{pmatrix}, \begin{pmatrix} x_1 & \cdots & x_i & x_{i+1} & \cdots & x_{i+j} \\ y_1 & \cdots & y_i & y_{i+1} & \cdots & y_{i+j} \\ z_1 & \cdots & z_i & z_{i+1} & \cdots & z_{i+j} \end{pmatrix}.$$

The later inequality is because of the irreducibility of $V(I_2(Y''))$ and $I_2(Y'') + \langle z_1, \ldots, z_i \rangle \subseteq I_2(Y)$. Since $\dim V(I_2(Y'')) = i+j+2$, we get the desired result. $\qquad\square$

In general, we can prove the following.

Proposition 4.3.3. *Let $Z_{i,j,m}$ be the matrix of indeterminants*

$$\begin{pmatrix} x_1 & \cdots & x_i & x_{i+1} & \cdots & x_{i+j} & x_{i+j+1} & \cdots & x_{i+j+m} \\ 0 & \cdots & 0 & y_1 & \cdots & y_j & y_{j+1} & \cdots & y_{j+m} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & z_1 & \cdots & z_m \end{pmatrix}.$$

Let $J = I_2(Z_{i,j,m})$. Then we have

(a) *If $m > 0, i = 0, j = 0$ then $\dim V(J) = m+2$*
(b) *If $m = 0, i = 0, j \ge 1$ then $\dim V(J) = j+1$*
(c) *If $m = 0, i \ge 1, j \ge 1$ then $\dim V(J) = i+j$*
(d) *If $m > 0, i = 0, j > 0$ then $\dim V(J) = m+j+1$*
(e) *If $m > 0, i = 1, j = 0$ then $\dim V(J) = m+2$*
(f) *If $m \ge 0, i \ge 1, j \ge 1$ then $\dim V(J) = i+j+m$*
(g) *If $m \ge 0, i \ge 2, j \ge 0$ then $\dim V(J) = i+j+m$.*

Proof. Parts (a), (b), (c), and (d) are easily follows from Proposition 2.4.1 and Lemmas 4.3.1 and 4.3.2.

(e) Checking the possibilities of x_1 (whether it is 0 or not), we can decompose $V(J)$ into the union of the affine space of dimension $m+1$ (corresponding to the coordinate ring $k[x_1, \ldots, x_{m+1}]$) and determinantal variety $V(I_2(Z_{0,0,m}))$ of dimension $m+2$; so the result follows.

(f) We first consider the case $i = 1, j = 1$. Analyzing x_1 as earlier, we have a decomposition of $V(J)$ as a union of the affine sapce of dimension $m+2$ (corresponding to the coordinate ring $k[x_1, \ldots, x_{m+2}]$) and the variety $V(I_2(Z_{0,1,m}))$ which is of dimension $m+2$ by Lemma 4.3.2. So $\dim V(J) = m+2$.

Now suppose that $i, j \ge 1$. Then note that $V(J) \subset V(I_2(Z_{1,1,i+j+m-2}))$ which implies that $\dim V(J) \le i+j+m$ by the previous case. As $V(J)$ contains the affine space corresponding to the coordinate ring $k[x_1, \ldots, x_{i+j+m}]$, we obtain that $\dim V(J) = i+j+m$.

(g) Similar to the previous part, we have $V(J) \subset V(I_2(Z_{i-1,j+1,m}))$ so that $\dim V(J) = i+j+m$ for the same reason as above. $\qquad\square$

Remark 4.3.4. Let n_0, n_1 be respectively the numbers of rows and columns containing zeros of $Z_{i,j,m}$. Then we set $N_{i,j,m} = \min(n_0, n_1)$. Observe that $N_{i,j,m} = 0, 1$ or 2. Moreover, we have

- $N_{i,j,m} = 0$ $\Leftrightarrow$ Condition in part (a),
- $N_{i,j,m} = 1$ $\Leftrightarrow$ Conditions in (b), (d), and (e),
- $N_{i,j,m} = 2$ $\Leftrightarrow$ Conditions in (c), (f), and (g).

Hence Proposition 4.3.3 can be rewritten as follows.

Theorem 4.3.5. *Let $Z_{i,j,m}$ be the matrix of indeterminants*

$$\begin{pmatrix} x_1 & \cdots & x_i & x_{i+1} & \cdots & x_{i+j} & x_{i+j+1} & \cdots & x_{i+j+m} \\ 0 & \cdots & 0 & y_1 & \cdots & y_j & y_{j+1} & \cdots & y_{j+m} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & z_1 & \cdots & z_m \end{pmatrix}.$$

Then we have $\dim V(I_2(Z_{i,j,m})) = i + j + m + 2 - N_{i,j,m}$.

It will be nice if we can generalize this result to larger matrices of indeterminants.

4.4. **Main Theorem.** We can now compute the dimension of the mixed commuting variety $C_{i,j,m}$. If $i = 0$, then the answer is obtained from Proposition 4.1.1. So we assume that $i \geq 1$.

Theorem 4.4.1. *Keep the notations from Remark 4.3.4. For each $i \geq 1$, we have*

$$\dim C_{i,j,m} = 2(i + j + m) + 4 - N_{i-1,j,m}.$$

Proof. Let first consider the decomposition

(3) $$C_{i,j,m} = G \cdot (v_{\text{sub}}, D_{i-1,j,m}) \cup 0 \times C_{i-1,j,m}$$

where $D_{i-1,j,m} = C(\underbrace{z_{\text{sub}} \cap \overline{\mathcal{O}_{\text{sub}}}, \ldots, z_{\text{sub}} \cap \overline{\mathcal{O}_{\text{sub}}}}_{i-1 \text{ times}}, \underbrace{z_{\text{sub}} \cap \mathcal{N}, \ldots, z_{\text{sub}} \cap \mathcal{N}}_{j \text{ times}}, \underbrace{z_{\text{sub}}, \ldots, z_{\text{sub}}}_{m \text{ times}})$. By Proposition 4.2.1, we can further decompose

$$D_{i-1,j,m} = C(\underbrace{V_1 \cup V_2, \ldots, V_1 \cup V_2}_{i-1 \text{ times}}, \underbrace{z_{\text{sub}} \cap \mathcal{N}, \ldots, z_{\text{sub}} \cap \mathcal{N}}_{j \text{ times}}, \underbrace{z_{\text{sub}}, \ldots, z_{\text{sub}}}_{m \text{ times}})$$

$$= C(\underbrace{V_1, \ldots, V_1}_{i-1 \text{ times}}, \underbrace{z_{\text{sub}} \cap \mathcal{N}, \ldots, z_{\text{sub}} \cap \mathcal{N}}_{j \text{ times}}, \underbrace{z_{\text{sub}}, \ldots, z_{\text{sub}}}_{m \text{ times}}) \cup$$

$$\cup C(\underbrace{V_2, \ldots, V_2}_{i-1 \text{ times}}, \underbrace{z_{\text{sub}} \cap \mathcal{N}, \ldots, z_{\text{sub}} \cap \mathcal{N}}_{j \text{ times}}, \underbrace{z_{\text{sub}}, \ldots, z_{\text{sub}}}_{m \text{ times}}).$$

Analyzing the commutators of V_1 or V_2 with $z_{\text{sub}} \cap \mathcal{N}$ and z_{sub}, we have the following identities

$$C(\underbrace{V_1, \ldots, V_1}_{i-1 \text{ times}}, \underbrace{z_{\text{sub}} \cap \mathcal{N}, \ldots, z_{\text{sub}} \cap \mathcal{N}}_{j \text{ times}}, \underbrace{z_{\text{sub}}, \ldots, z_{\text{sub}}}_{m \text{ times}}) = V(I) \times k^{i+j+m-1}$$

$$C(\underbrace{V_2, \ldots, V_2}_{i-1 \text{ times}}, \underbrace{z_{\text{sub}} \cap \mathcal{N}, \ldots, z_{\text{sub}} \cap \mathcal{N}}_{j \text{ times}}, \underbrace{z_{\text{sub}}, \ldots, z_{\text{sub}}}_{m \text{ times}}) = V(J) \times k^{i+j+m-1}$$

where the affine space $k^{i+j+m-1}$ is from the freeness of $y_1, \ldots, y_{i+j+m-1}$, I and J are respectively the ideals generated by 2×2 minors of the following matrices

$$\begin{pmatrix} z_1 & \cdots & z_{i-1} & z_i & \cdots & z_{i+j-1} & z_{i+j} & \cdots & z_{i+j+m-1} \\ 0 & \cdots & 0 & t_1 & \cdots & t_j & t_{j+1} & \cdots & t_{j+m} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & x_1 & \cdots & x_m \end{pmatrix}$$

$$\begin{pmatrix} t_1 & \cdots & t_{i-1} & t_i & \cdots & t_{i+j-1} & t_{i+j} & \cdots & t_{i+j+m-1} \\ 0 & \cdots & 0 & z_1 & \cdots & z_j & z_{j+1} & \cdots & z_{j+m} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & x_1 & \cdots & x_m \end{pmatrix}.$$

Proposition 4.3.3 and Theorem 4.3.5 give us

$$\dim V(I) = \dim V(J) = i + j + m + 1 - N_{i-1,j,m}$$

so that
$$\dim D_{i-1,j,m} = 2(i+j+m) - N_{i-1,j,m}.$$

Hence we have
$$\dim G \cdot (v_{\mathrm{sub}}, D_{i-1,j,m}) = 4 + \dim D_{i-1,j,m} = 2(i+j+m) + 4 - N_{i-1,j,m}.$$

The remaining is to prove that $\dim 0 \times C_{i-1,j,m} \leq 2(i+j+m) + 4 - N_{i-1,j,m}$. Indeed, if $i = 1$, then it equals to $2(j+m) + 4$ (by Proposition 4.1.1) which is $\leq 2(1+j+m) + 4 - N_{i-1,j,m}$ since $N_{i-1,j,m} \leq 2$. For $i > 1$, we have by induction that
$$\dim C_{i-1,j,m} = 2(i-1+j+m) + 4 - N_{i-2,j,m} \leq 2(i+j+m) + 2 \leq 2(i+j+m) + 4 - N_{i-1,j,m}.$$

Finally, we have shown that $\dim C_{i,j,m} = 2(i+j+m) + 4 - N_{i-1,j,m}$ as desired. $\qquad \square$

Next we show that mixed commuting varieties are usually not irreducible. We start with a lemma.

Lemma 4.4.2. *For each $i \geq 2$, the variety $C_{i,0,0} = C_i(\overline{\mathcal{O}_{\mathrm{sub}}})$ is reducible of dimension $2r + 2$.*

Proof. This is just a corollary of [N, Theorem 7.2.3]. In particular, we have the following irreducible decomposition
$$C_{i,0,0} = \overline{G \cdot (v_{\mathrm{sub}}, V_1, \ldots, V_1)} \cup \overline{G \cdot (v_{\mathrm{sub}}, V_2, \ldots, V_2)}$$
where V_1 and V_2 are defined in Proposition 4.2.1. $\qquad \square$

Theorem 4.4.3. *For each $i, j, m \geq 0$, the mixed commuting variety $C_{i,j,m}$ is irreducible if and only if i, j, m satisfy one of the following conditions:*

    (1) $i = j = 0$,
    (2) $i = m = 0$,
    (3) $i = 1$, $j = m = 0$.

Proof. The conditions (1), (2), and (3) in the theorem are equivalent to the cases in which the variety $C_{i,j,m}$ is either $C_m(\mathfrak{sl}_3)$, or $C_j(\mathcal{N})$, or $\overline{\mathcal{O}_{\mathrm{sub}}}$. It is known that these varieties are irreducible. Indeed, the variety $C_j(\mathcal{N})$ is irreducible by Theorem 7.1.2 in [N], and the variety $C_m(\mathfrak{sl}_3)$ is irreducible by Theorem 4.2.1 in [N] and the fact that $C_m(\mathfrak{gl}_3)$ is irreducible [KN][Gu]. From the decomposition (3) in Theorem 4.4.1, we have
$$C_{i,j,m} = \overline{G \cdot (v_{\mathrm{sub}}, D_{i-1,j,m})} \cup 0 \times C_{i-1,j,m}$$
where $\overline{G \cdot (v_{\mathrm{sub}}, D_{i-1,j,m})} = G \cdot (v_{\mathrm{sub}}, D_{i-1,j,m}) \cup \{0\}$. So if $i \geq 1$ and $j \neq 0$ or $m \neq 0$, we always have $C_{i,j,m}$ is reducible. The case $i > 1$ and $j = m = 0$ was proved in the previous lemma. On the other hand, Proposition 4.1.1 shows that $C_{i,j,m}$ is reducible in the case $i = 0$ and $j, m \geq 1$. $\qquad \square$

Remark 4.4.4. This result also indicates that mixed commuting varieties are rarely normal as all irreducible components contain the origin. In other words, if a mixed commuting variety is reducible, it is not normal. This behavor is analogous with that of mixed commuting varieties over $\mathfrak{sl}_2$ and it nullcone in [N, Proposition 6.1.1].

5. Applications to support varieties for Frobenius kernels

5.1. **Support varieties.** Let G be a simple algebraic group defined over k (we assume that $p \geq 3$ in this section). For each $r \geq 1$, let
$$\mathrm{H}^{\bullet}(G_r, k) = \bigoplus_{i \geq 0} \mathrm{H}^i(G_r, k) \quad , \quad \mathrm{H}^{2\bullet}(G_r, k) = \bigoplus_{i \geq 0} \mathrm{H}^{2i}(G_r, k).$$

Under the cup product, $\mathrm{H}^{2\bullet}(G_r, k)$ is a commutative ring. Given a finite dimensional G-module M, we consider $\mathrm{Ext}^{\bullet}_{G_r}(M, M)$ a $\mathrm{H}^{2\bullet}(G_r, k)$-module with the action induced by the cup product. Then the support variety of M, denoted by $V_{G_r}(M)$, is the annihilator of $\mathrm{Ext}^{\bullet}_{G_r}(M, M)$ in the

ring $\mathrm{H}^{2\bullet}(G_r, k)$. Note that $V_{G_r}(k) = \mathrm{Spec}\ \mathrm{H}^{2\bullet}(G_r, k)_{\mathrm{red}} = C_r(\mathcal{N})$ when the characteristic p is big enough [SFB1].

Let λ is a weight in X and set

$$\Phi_{\lambda,p} = \{\beta \in \Phi \mid (\lambda + \rho, \beta^\vee) \in p\mathbb{Z}\}.$$

Suppose that the prime p is good for G. We recall that λ is called p-regular if $\Phi_{\lambda,p} = \emptyset$, otherwise it's called p-singular. The following result classifies the behaviors of support varieties for $L(\lambda)$ up to its regularity in the case of simple algebraic groups of rank 2.

Proposition 5.1.1. [NPV, Corollary 6.6.1] *Let G be a simple algebraic group of rank 2 and p good.*
  (a) *If λ is a p-singular weight, then $V_{G_1}(L(\lambda)) = \overline{\mathcal{O}_{\mathrm{sub}}}$.*
  (b) *If λ is a p-regular weight, then $V_{G_1}(L(\lambda)) = V_{G_1}(k) = \mathcal{N}$.*

Proposition 5.1.2. [So, Theorem 3.2] *Let G be a classical simple algebraic group. Suppose that $p > hc^2$. Let λ be a weight with $\lambda = \lambda_0 + p\lambda_1 + \cdots + p^q\lambda_q$, $\lambda_i \in X_1(T)$. Then for each $r \geq 1$, we have*

$$V_{G_r}(L(\lambda)) = \{(\beta_0, \ldots, \beta_{r-1}) \in C_r(\mathcal{N}) \mid \beta_i \in V_{G_1}(L(\lambda_{r-i-1}))\}.$$

Now suppose that G is of type A_2 and $p \geq 7$ (i.e., $p > hc$). Let λ be a weight such that $\lambda = \lambda_0 + p\lambda_1 + \cdots + p^q\lambda_q$ with $\lambda_i \in X_1(T)$. Let a_λ, b_λ be the number of singular weights and regular weights respectively in $\{\lambda_1, \ldots, \lambda_q\}$. Then by the propositions above, we have for each $r \geq 1$

$$V_{G_r}(L(\lambda)) = C(\underbrace{\overline{\mathcal{O}_{\mathrm{sub}}}, \ldots, \overline{\mathcal{O}_{\mathrm{sub}}}}_{a_\lambda \text{ times}}, \underbrace{\mathcal{N}, \ldots, \mathcal{N}}_{b_\lambda \text{ times}}) = C_{a_\lambda, b_\lambda, 0}.$$

This is a special case of mixed commuting varieties appearing in previous sections. Hence we know the dimensional and irreducible behaviors of the support variety $V_{G_r}(L(\lambda))$.

Theorem 5.1.3. *Let G be of type A_2 and $p > 6$. Suppose λ is a weight in X. Then for each $r \geq 1$, the support variety $V_{G_r}(L(\lambda))$ can be identified with the mixed commuting variety $C_{a_\lambda, b_\lambda, 0}$. Furthermore,*

$$\dim V_{G_r}(L(\lambda)) = \begin{cases} 2b_\lambda + 4 & \text{if } a_\lambda = 0, \\ 2(a_\lambda + b_\lambda) + 3 & \text{if } a_\lambda = 1, \\ 2(a_\lambda + b_\lambda) + 2 & \text{if } a_\lambda > 1. \end{cases}$$

Theorem 5.1.4. *Under the same assumption as in the previous theorem, for each $r \geq 2$, the support variety $V_{G_r}(L(\lambda))$ is irreducible if and only if $a_\lambda = 0$. In other words, $V_{G_r}(L(\lambda))$ is irreducible if and only if there is no singular weight in the decomposition of λ.*

Proof. It immediately follows from Theorem 4.4.3. $\square$

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[2]See [So, Section 3] for the value of c for each type of G

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